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Asymptotic probability density functions in turbulence.

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Abstract

A formalism is presented to obtain closed evolution equations for asymptotic probability distribution functions of turbulence magnitudes. The formalism is derived for a generic evolution equation, so that the final result can be easily applied to rather general problems. Although the approximation involved cannot be ascertained a priori, we show that application of the formalism to well known problems gives the correct results.

I. Introduction

Probability distribution functions (pdf's) are powerful tools for the experimental and theoretical study of turbulence. Experimental pdf's can be obtained rather directly from the measurements of different magnitudes, and are at the same time a very convenient tool for the representation of such results. From the theoretical point of view, the equations for the evolution of pdf's can be readily obtained from the basic dynamical equations, although generally not in closed form. Closure requires the determination of conditional averages in terms of the pdf itself, which sometimes can be done based on additional information given by experiments,^{1,2} and numerical simulations.^{3,4} Closures of this kind have allowed to derive very important results directly from the original dynamical equations, such as the limiting and time dependent pdf of passive^{5,6} and non-passive⁷ scalars. We present here a procedure for closure of the pdf equation, at least of its asymptotic form, that can be applied in a systematic way to general systems. It is important to mention that one can only show that the model is applicable if a reasonable conjecture is valid. We show that the model gives correct results in various systems whose behavior is known. The advantage is that it is very easy to apply and so it is valuable to attack new problems for which no useful information exists.

II. Formalism

The basic idea is most easily introduced considering a generic scalar field $\phi(x, t)$, where t is the time, and x denotes the spatial coordinates. The field ϕ satisfies a generic evolution equation of the form

$$\frac{\partial \phi}{\partial t} = N[\phi, x] + f(x, t), \quad (1)$$

in which $N[\phi]$ is a spatial functional of $\phi(x, t)$ evaluated at the actual time t , and f is a stochastic forcing introduced to allow eventual consideration of statistically stationary pdf's. $N[\phi]$ contains in general linear and non-linear terms, and we restrict the non-linear terms to entire powers of ϕ , not necessarily local. The induced probability distribution of a given functional of ϕ , $G[\phi]$, can be conveniently represented as

$$P(\xi) = \langle \delta(G[\phi] - \xi) \rangle, \quad (2)$$

in which $\langle \dots \rangle$ means average over ensembles of realizations, and δ is Dirac's delta function. The evolution equation of $P(\xi)$ is then determined from Eq. (1) as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \xi} \left[\left\langle \frac{\delta G}{\delta \phi(x_1)} * [N[\phi, x_1] + f(x_1, t)] \right| \xi \right] P, \quad (3)$$

where $\langle \dots | \xi \rangle$ represents average conditioned on $G[\phi] = \xi$, and the asteric indicates integration over repeated variables. We will consider only functionals $G[\phi]$ that are linear in ϕ , so that $(\delta G / \delta \phi) *$ will be an operator independent of ϕ which we denote as

$$\frac{\delta G}{\delta \phi(x_1)} \equiv h(x_1).$$

Besides, as $h(x_1)$ is independent of ϕ , only $\langle N[\phi] + f | \xi \rangle$ needs to be evaluated.

To evaluate $\langle N[\phi] | \xi \rangle$ let us consider the simplest linear term,

$$\Gamma_1(y, \xi) \equiv \langle \phi(y) | h(x_1) * \phi(x_1) = \xi \rangle$$

where y is an arbitrary spatial position and, from now on, no explicit indication of the time is made. The exact, formal expression of $\Gamma_1(y, \xi)$ can be written as

$$\Gamma_1(y, \xi) = \frac{\int \mathcal{F}[\phi] \phi(y) \delta(h(x_1) * \phi(x_1) - \xi) \mathcal{D}\phi}{\int \mathcal{F}[\phi] \delta(h(x_1) * \phi(x_1) - \xi) \mathcal{D}\phi}. \quad (4)$$

Here Dirac's delta function δ is used to select only those fields that satisfy the condition $h(x_1) * \phi(x_1) = \xi$, $\mathcal{F}[\phi]$ is the probability density functional (pdf) of the field ϕ , and the integrations are meant to be functional integrations over fields ϕ , with an appropriate measure $\mathcal{D}\phi$. In principle, given the system (1) a Martin-Siggia-Rose lagrangian can be determined in terms of which to express $\mathcal{F}[\phi]$. Once this is done, solutions of (4) can be obtained in terms of perturbative expansions, or non-perturbative approaches such as rapid descent type of integrations around significant field configurations. We follow here the perturbative approach, setting up an infinite series that represents (4) in a formally exact manner, and then dividing this series into two infinite other series, such that one of them is term by term much larger than the other in the limit of large ξ . We keep only these larger terms and sum the resulting infinite series.

To proceed we write both integrands appearing in (4) as series involving Gaussian functionals. For this we write

$$\mathcal{F}[\phi] = \mathcal{F}_0[\phi] \mathcal{G}[\phi], \quad (5)$$

where

$$\mathcal{F}_0[\phi] = Z_0^{-1} \exp[-\kappa(x_1) * \phi(x_1) - \frac{1}{2}\sigma(x_1, x_2) * \phi(x_1)\phi(x_2)], \quad (6)$$

is a Gaussian functional, with $\kappa(x)$ and $\sigma(x, x')$ functions to be determined, and Z_0 a normalization constant ensuring that $\int \mathcal{F}_0[\phi] \mathcal{D}\phi = 1$. $\mathcal{G}[\phi]$ is in general a non-Gaussian functional factor defined by the very expression (5), satisfying also the normalization condition $\int \mathcal{F}[\phi] \mathcal{D}\phi = 1$. We further expand $\mathcal{G}[\phi]$ as a functional Taylor series

$$\mathcal{G}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} K(x_1, \dots, x_n) * \phi(x_1) \dots \phi(x_n), \quad (7)$$

where the $n = 0$ term is to be understood as constant.

We turn now to the evaluation of (4). For this, note that the denominator of this expression is the pdf (2), $P(\xi)$, which, using the Fourier representation of Dirac's delta function, can be written as

$$\begin{aligned} P(\xi) &= \int \mathcal{F}[\phi] \delta(h(x_1) * \phi(x_1) - \xi) \mathcal{D}\phi = \\ &= (2\pi)^{-1} \int dk \exp(-ik\xi) \\ &= \int \mathcal{F}_0[\phi] \mathcal{G}[\phi] \exp[ikh(x_1) * \phi(x_1)] \mathcal{D}\phi. \end{aligned} \quad (8)$$

It is now useful to introduce an auxiliary functional given by

$$\begin{aligned} \mathcal{A}[J] &= (2\pi)^{-1} \int dk \exp(-ik\xi) \\ &= \int \mathcal{F}_0[\phi] \exp[J(x_1) * \phi(x_1) + ikh(x_1) * \phi(x_1)] \mathcal{D}\phi. \end{aligned} \quad (9)$$

\mathcal{A} plays the role of a potential from which (8) and the numerator of (4) can be obtained by differentiation as it is immediately seen from (9) that, for instance,

$$\frac{\delta \mathcal{A}}{\delta J(x_k)} = \phi(x_k) \mathcal{A}[J], \quad (10)$$

and so, successive derivatives allow to produce the factors ϕ appearing in the series of $\mathcal{G}[\phi]$ (see (7)). In this way, we write the denominator and numerator

of (4) respectively as

$$P(\xi) = \sum_n \frac{1}{n!} K(x_1, \dots, x_n) * \mathcal{A}^{(n)} \Big|_{J=0}, \quad (11)$$

and

$$\Gamma_1(y, \xi) P(\xi) = \sum_n \frac{1}{n!} K(x_1, \dots, x_n) * \frac{\delta \mathcal{A}^{(n)}}{\delta J(y)} \Big|_{J=0}, \quad (12)$$

where

$$\mathcal{A}^{(n)} \equiv \frac{\delta^n \mathcal{A}}{\delta J(x_1) \dots \delta J(x_n)}. \quad (13)$$

Evaluating the Gaussian integrals in (8) and (9) we obtain

$$\mathcal{A}[J] = N \exp(-\Delta^{-1} \xi^2 / 2) \exp W[J - \kappa], \quad (14)$$

where N is a factor independent of J and ξ , $\Delta = \Sigma(x_1, x_2) * h(x_1)h(x_2)$, and

$$W[X] = 1/2 [\Sigma(x_1, x_2) * X(x_1)X(x_2) - \Delta^{-1}b^2 + 2\Delta^{-1}b\xi], \quad (15)$$

with $b[X] = \Sigma(x_1, x_2) * h(x_1)X(x_2)$ a linear functional of X , and Σ is the inverse of σ in the sense that

$$\Sigma(x_1, x_2) * \sigma(x_2, x_3) = \sigma(x_1, x_2) * \Sigma(x_2, x_3) = \delta(x_1 - x_3). \quad (16)$$

In this way, (4) can be calculated by explicit differentiation through (11) and (12), the series so obtained being formally equivalent to the original integrals.

So far we have only set up a generic series expansion in analogy with usual methods, and to obtain useful results this series needs to be evaluated at least approximately. We assume no intrinsic small parameter, but rather consider the condition of large values of $|\xi|$. Differentiating (14) one can write

$$\frac{\delta \mathcal{A}}{\delta J(y)} = \frac{\delta W}{\delta J(y)} \mathcal{A}, \quad (17)$$

We will not write at this point the explicit expression of $\delta W / \delta J(y)$ but just point out its pertinent properties:

- i) It is a linear functional of J and a linear function of ξ .
- ii) Every derivative of $\delta W / \delta J(y)$ with respect to J deletes a factor ξ . In contrast, every derivative of \mathcal{A} brings up a factor ξ .

With these considerations we now examine a generic derivative in the sum over n in (12) and write in a notation symbolizing the n -order derivative of the product in the r.h.s. of (17)

$$\frac{\delta \mathcal{A}^{(n)}}{\delta J(y)} = \frac{\delta^n (\delta \mathcal{A} / \delta J(y))}{\delta J(x_1) \dots \delta J(x_n)} = \sum_{p=0}^n \binom{n}{p} \mathcal{A}^{(n-p)} \frac{\delta^p}{\delta J^p} \left[\frac{\delta W}{\delta J(y)} \right]. \quad (18)$$

By property i) the sum over p runs only from 0 to 1, so that we can write the expansion (12) as

$$\begin{aligned} \Gamma_1(y, \xi) P(\xi) &= \left. \frac{\delta W}{\delta J(y)} \right|_{J=0} \sum_n \frac{1}{n!} K(x_1, \dots, x_n) * \mathcal{A}^{(n)} \Big|_{J=0} \\ &+ \sum_n \left\{ \frac{1}{(n-1)!} K(x_1, \dots, x_n) * \right. \\ &\quad \left. \left[\mathcal{A}^{(n-1)} \frac{\delta^2 W}{\delta J(x_n) \delta J(y)} \right]_{J=0} \right\} \end{aligned}$$

By property ii) each “ n ” term in the first series is two orders in ξ higher than the corresponding term in the second series and, besides, according to expression (11) the first series is equal to $P(\xi)$. In this way, the sum of the second series must be equal to $(\Gamma_1(y, \xi) - \delta W / \delta J(y)|_{J=0}) P(\xi)$, which, by construction, is a finite magnitude. We have thus two convergent series, one of which is term by term much larger than the other in the limit $|\xi| \rightarrow \infty$. Although this is not sufficient to ensure that the sum of the first series dominates over the sum of the second series, we obtain a plausible, simple model if we assume it and write in the large $|\xi|$ limit

$$\Gamma_1(y, \xi) \rightarrow \delta W / \delta J(y)|_{J=0} = \Delta^{-1} \xi \Sigma(y, x_1) * h(x_1), \quad (19)$$

where $\Delta = \Sigma(x_1, x_2) * h(x_1) h(x_2)$. The function κ does not appear in (19) because it contributes only to lower order terms in $\delta W / \delta J(y)|_{J=0}$.

The same argument applies to a generic non-linear term as

$$\Gamma_m(y_1, \dots, y_m, \xi) \equiv \langle \phi(y_1) \dots \phi(y_m) | h(x_1) * \phi(x_1) = \xi \rangle, \quad (20)$$

which can be obtained from (compare with Eq. (12))

$$\Gamma_m P(\xi) = \sum_n \frac{1}{n!} K(x_1, \dots, x_n) * \left. \frac{\delta^m \mathcal{A}^{(n)}}{\delta J(y_1) \dots \delta J(y_m)} \right|_{J=0}.$$

Properties i) and ii) allow this series to be split into $m + 1$ series, such that m of them are term by term much smaller (by factors $\xi^{-2}, \dots, \xi^{-2m}$) than the “dominant” series

$$\left. \frac{\delta^m W}{\delta J(y_1) \dots \delta J(y_m)} \right|_{J=0} \sum_n \frac{1}{n!} K(x_1, \dots, x_n) * \mathcal{A}^{(n)} \Big|_{J=0}.$$

Again, by comparison with expression (11), this series is convergent, so that the remaining series is also convergent. We thus model Γ_m for large values of $|\xi|$ as

$$\begin{aligned} \Gamma_m(y_1, \dots, y_m, \xi) &\rightarrow \left. \frac{\delta^m W}{\delta J(y_1) \dots \delta J(y_m)} \right|_{J=0} \\ &= \Delta^{-m} \xi^m \Sigma(y_1, x_1) * \\ &\quad h(x_1) \dots \Sigma(y_m, x_m) * h(x_m). \end{aligned} \quad (21)$$

An important consistency check is that contraction of (21) with $h(y_1) \dots h(y_m)$ satisfies the exact identity

$$\Gamma_m(y_1, \dots, y_m, \xi) * h(y_1) \dots h(y_m) = \xi^m,$$

for any $h(x_1)$.

So far no conditions have been imposed on the functions $\kappa(x)$ and $\sigma(x, y)$. The only restriction is that both, $\mathcal{F}_0[\phi]$ and $\mathcal{F}[\phi]$ are pdf's of ϕ , so that they are normalized to unity for the same measure $\mathcal{D}\phi$

$$\int \mathcal{F}_0[\phi] \mathcal{D}\phi = \int \mathcal{F}[\phi] \mathcal{D}\phi = 1,$$

and that, of course, $\mathcal{F}[\phi]$ is independent of $\kappa(x)$ and $\sigma(x, y)$. From these conditions and the expression (6) of $\mathcal{F}_0[\phi]$ we easily obtain

$$\frac{\delta \ln \mathcal{G}}{\delta \kappa(x)} = -\frac{\delta \ln \mathcal{F}_0}{\delta \kappa(x)} = \phi(x) - \langle \phi(x) \rangle_0, \quad (22a)$$

$$\frac{\delta \ln \mathcal{G}}{\delta \sigma(x, y)} = -\frac{\delta \ln \mathcal{F}_0}{\delta \sigma(x, y)} = \frac{1}{2} [\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle_0], \quad (22b)$$

where $\langle \dots \rangle_0$ stands for the average using \mathcal{F}_0 . To choose the functions $\kappa(x)$ and $\sigma(x, y)$ consider that we have modeled the large ξ asymptotics of the Γ_m terms as the sum of a “dominant” series in the sense described above.

Since the sums involve infinite series, the term by term dominance is not sufficient to ensure dominance of the sum, and that is why what we obtain is an approximation whose validity cannot be ascertained in general. However, we expect the approximation to be better the faster the series (7) converge, so that fewer terms are required in the sums to approximate the functions at a given value of ξ . With only the freedom to choose $\kappa(x)$ and $\sigma(x, y)$ we can then expect better results if these functions are chosen so as to keep $\mathcal{G}[\phi]$ as small as possible in some averaged sense. Taking into account that \mathcal{G} is a definite positive magnitude, we could minimize $\int \mathcal{F} \mathcal{G} \mathcal{D}\phi$, or $\int \mathcal{F} \mathcal{G}^2 \mathcal{D}\phi$, or $\int \mathcal{F} \ln \mathcal{G} \mathcal{D}\phi$, etc.. Of these possible choices, the one that weights preferentially the small values of \mathcal{G} is $\int \mathcal{F} \ln \mathcal{G} \mathcal{D}\phi$, which, using relations (22), leads immediately to ($\langle \dots \rangle$ represents the average using \mathcal{F} ; that is, the true average)

$$\langle \phi(x) \rangle = \langle \phi(x) \rangle_0 = -\Sigma(x, x_1) * \kappa(x_1), \quad (23)$$

$$\begin{aligned} \langle \phi(x) \phi(y) \rangle &= \langle \phi(x) \phi(y) \rangle_0 = \\ &\Sigma(x, y) + \kappa(x_1) \kappa(y_1) * \Sigma(x_1, x) \Sigma(y_1, y). \end{aligned} \quad (24)$$

These equations are easily inverted to determine the functions $\Sigma = \sigma^{-1}$ and κ as

$$\Sigma(x, y) = \langle \phi(x) \phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle. \quad (25)$$

$$\kappa(x) = -\sigma(x, x_1) * \langle \phi(x_1) \rangle, \quad (26)$$

We then summarize the model for the asymptotic averages as

$$\begin{aligned} \Gamma_m(y_1, \dots, y_m, \xi) &\rightarrow \xi^m [\Sigma(x_0, x'_0) * h(x_0) h(x'_0)]^{-m} \\ &\Sigma(y_1, x_1) * h(x_1) \dots \Sigma(y_m, x_m) * h(x_m), \end{aligned} \quad (27)$$

with $\Sigma(x, y)$ given by Eq. (25).

Finally, the conditional average of the stochastic Gaussian force of zero mean can be obtained explicitly for the case of very short lived force correlation of the form¹⁸

$$\langle f(x, t) f(x', t') \rangle = \delta(t - t') F(x - x'),$$

and is given in our notation as

$$\langle f(x, t) | \xi \rangle = -\frac{1}{2} \frac{\partial p}{\partial \xi} h(x_1) * F(x - x_1). \quad (28)$$

III. Applications

A. Temperature diffusion term

As a first simple check, we consider the average related to temperature diffusion for which measurements exist⁸

$$s(\xi) \equiv \langle \nabla^2 \phi | \phi(x) = \xi \rangle.$$

It corresponds to $h(x_1) = \delta(x_1 - x)$, and can be modeled according to Eq. (27) as

$$s(\xi) = \nabla_y^2 \Gamma_1(y, \xi) \big|_{y=x} \rightarrow \xi \frac{\langle \phi(x) \nabla^2 \phi \rangle - \langle \phi(x) \rangle \nabla^2 \langle \phi(x) \rangle}{\langle \phi^2(x) \rangle - \langle \phi(x) \rangle^2}. \quad (29)$$

It is convenient to take ϕ as a non-dimensional field of zero mean and unit variance, defined from the physical temperature field $T(x)$ as

$$\phi(x) = \frac{T(x) - \langle T(x) \rangle}{\sigma_T}, \quad (30)$$

where σ_T is the variance of T . The measured non-dimensional conditional average is defined as

$$r(\xi) \equiv \langle (|\nabla \phi|^2)^{-1} s(\xi) \rangle, \quad (31)$$

which, from (29) and the fact that by definition the variance of ϕ is constant, reduces to

$$r(\xi) \rightarrow -\xi. \quad (32)$$

This simple result of the model has been experimentally seen to hold for a wide range of values of ξ .²

B. Forced Burgers turbulence

For a randomly forced Burgers flow, the velocity $u(x, t)$ satisfies⁹

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (33)$$

where ν is the kinematic viscosity, $f(x, t)$ is a random force with Gaussian distribution of zero mean and variance given by

$$\langle f(x, t) f(x', t') \rangle = \delta(t - t') F(x - x'), \quad (34)$$

with F an even function of its argument which decays sufficiently fast for $|x - x'|$ larger than a correlation length L_c . Besides, $F(0) = 2\varepsilon'$, where ε' is the rate of injection of energy density. Eq. (33) with forcing defined by (34) has been extensively studied using different, powerful theoretical methods^{10,11,12,13,14,15} and numerical simulations.^{16,17}

1. Pdf of velocity difference

The statistical magnitude to be considered first is the instantaneous velocity difference across a separation r , $G[u] = u(x + r, t) - u(x, t)$. The evolution equation for the corresponding pdf $P(\xi, r, t)$ is, from Eqs. (2) and (28),

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \xi} (\langle A | \xi, r \rangle P) + [F(0) - F(r)] \frac{\partial^2 P}{\partial \xi^2}, \quad (35)$$

where

$$A = \left[u(\eta, t) \frac{\partial}{\partial \eta} u(\eta, t) - \nu \frac{\partial^2}{\partial \eta^2} u(\eta, t) \right] * h(\eta), \quad (36)$$

$h(\eta) = \delta G / \delta u(\eta) = \delta(x + r - \eta) - \delta(x - \eta)$, with $\langle \dots | \xi, r \rangle$, as above, a short notation for $\langle \dots | u(x + r, t) - u(x, t) = \xi \rangle$. Finally, statistical spatial homogeneity allows to simplify the expression of A to write (35) as

$$\begin{aligned} \frac{\partial P}{\partial t} = & -2 \frac{\partial}{\partial \xi} \left(\left\langle u \frac{\partial u}{\partial x} \middle| \xi, r \right\rangle P \right) + \\ & 2\nu \frac{\partial}{\partial \xi} \left(\left\langle \frac{\partial^2 u}{\partial x^2} \middle| \xi, r \right\rangle P \right) + \\ & [F(0) - F(r)] \frac{\partial^2 P}{\partial \xi^2}. \end{aligned} \quad (37)$$

Using the model (27) we now evaluate the asymptotic expressions of the conditional averages appearing in this equation, which are conveniently ex-

pressed as

$$2 \left\langle u \frac{\partial u}{\partial x} | \xi, r \right\rangle = \frac{\partial}{\partial R} \left\langle u^2(R) | \xi, r \right\rangle \Big|_{R=0}, \quad (38)$$

$$\nu \left\langle \frac{\partial^2 u}{\partial x^2} | \xi, r \right\rangle = \nu \frac{\partial^2}{\partial R^2} \left\langle u(R) | \xi, r \right\rangle \Big|_{R=0}. \quad (39)$$

Using expression (27) with $m = 1, 2$ one obtains asymptotically

$$\langle u(R) | \xi, r \rangle \rightarrow \frac{\xi}{2S_2(r)} [S_2(R) - S_2(r - R)]. \quad (40)$$

$$\begin{aligned} \langle u^2(R) | \xi, r \rangle \rightarrow \frac{\xi^2}{4S_2^2(r)} [S_2^2(r - R) + \\ S_2^2(R) - 2 S_2(R) S_2(r - R)], \end{aligned} \quad (41)$$

where it was used that in the spatially homogeneous case considered one can write $\Sigma(x, x') = \langle u(x)u(x') \rangle = \langle u(x)u(x) \rangle - 1/2 S_2(x - x')$, with $S_2(x - x') = \langle (u(x) - u(x'))^2 \rangle$ the second order structure function. To evaluate the R derivatives appearing in (38) and (39) one needs to know the behavior of S_2 for small separation. This can be done because at very small scales, less than the dissipative scale set up by viscous effects, the velocity field is a smooth function of x and so one has^{19,20} $u(x + R) = u(x) + R\partial u/\partial x + O(R^2)$, which leads to $S_2(R) = R^2 \langle (\partial u/\partial x)^2 \rangle + O(R^3)$. With all this, (38) and (39) can be readily evaluated as

$$2 \left\langle u \frac{\partial u}{\partial x} | \xi, r \right\rangle \rightarrow -\frac{1}{2} \xi^2 \frac{\partial \ln S_2}{\partial r}, \quad (42)$$

$$\nu \left\langle \frac{\partial^2 u}{\partial x^2} | \xi, r \right\rangle \rightarrow \xi \frac{\varepsilon \Pi(r)}{S_2(r)}. \quad (43)$$

where it was used that $\nu \langle (\partial_x u)^2 \rangle = \varepsilon$, with ε the rate of dissipation of energy density, and where $\Pi(r) = 1 - \nu (\partial^2 S_2/\partial r^2)/(2\varepsilon)$. The asymptotic equation then reads

$$\begin{aligned} \frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial \ln S_2}{\partial r} \frac{\partial}{\partial \xi} (\xi^2 P) + \\ \frac{2\varepsilon \Pi}{S_2} \frac{\partial}{\partial \xi} (\xi P) + [F(0) - F(r)] \frac{\partial^2 P}{\partial \xi^2}. \end{aligned} \quad (44)$$

This equation is not strictly closed because the evaluation of $S_2(r) = \int \xi^2 P(\xi, r) d\xi$ requires the knowledge of P in the whole range of ξ , not only in the large ξ asymptotic where (44) holds. However, S_2 is a much simpler object to deal with, either theoretically or experimentally, than conditional averages. Besides, much and important information can be gathered without knowledge of its explicit expression as shown now. Let us consider now the stationary solutions of (44), in which case the rate of energy injection ε' equals the rate of energy dissipation ε . The forcing correlation is then written as

$$F(r) = 2\varepsilon\chi(r/L_c), \quad (45)$$

where χ is an even, dimensionless function that rapidly decays for arguments larger than one, also satisfying $\chi(0) = 1$. For r large compared to the forcing correlation length L_c one then has $F(r) \simeq 0$, and $S_2 \simeq 2\langle u^2 \rangle$, which is r independent. With these considerations, (44) immediately leads to the Gaussian solution $P(\xi) = \exp(-\xi^2/(2S_2))$, as it must since the two velocities involved are statistically independent for the r 's considered. Much more interesting is the situation $r \ll L_c$, in which case one can write $F(r) = 2\varepsilon[1 + \chi''(0)(r/L_c)^2/2 + O((r/L_c)^4)]$, with $\chi''(0) < 0$. If $\xi^2 P \rightarrow 0$ at large ξ , the immediate integral of (44) gives

$$\ln P = \frac{L_c^2}{r^2 \chi''(0)} \left[\frac{\xi^3}{6\varepsilon} \frac{\partial \ln S_2}{\partial r} + \frac{\xi^2 \Pi}{S_2} \right]. \quad (46)$$

If in the range of r considered $S_2(r)$ behaves as a power law, (in fact, $S_2(r) \sim r^2$ for r smaller than the dissipation scale, as seen above, and $S_2(r) \sim r$ in the inertial range²¹) we have $\partial \ln S_2 / \partial r \sim r^{-1}$. Taking into account that $\chi''(0) < 0$, it then results that the leading behavior for large ξ is $P \sim \exp[-L_c^2 \xi^3 / (\varepsilon r^3)]$. This solution is of course valid only for $\xi > 0$. For $\xi < 0$ then, $\xi^2 P$ cannot approach zero as $\xi \rightarrow -\infty$, and so P must behave as

$$P = \frac{C(r)}{\xi^2}, \quad (47)$$

where $C(r)$ is an unknown, positive function of r .

2. Pdf of velocity derivative

Let us consider now the pdf of velocity derivative at the origin, $G[u] = \partial u / \partial x|_{x=0}$, so that $h(\eta) = \delta G / \delta u(\eta) = -\delta'(\eta)$, where the prime indicates

derivative with respect to the argument. From Eqs. (2) and (28) we readily obtain

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{\partial}{\partial \xi} \left[\left(\xi^2 + \left\langle u \frac{\partial^2 u}{\partial x^2} \middle| \xi \right\rangle - \nu \left\langle \frac{\partial^3 u}{\partial x^3} \middle| \xi \right\rangle \right) P \right] \\ &\quad - \frac{1}{2} F''(0) \frac{\partial^2 P}{\partial \xi^2}. \end{aligned}$$

We now write, using expression (27) with $m = 2$,

$$\begin{aligned} \left\langle u \frac{\partial^2 u}{\partial x^2} \middle| \xi \right\rangle &= \frac{\partial^2}{\partial x^2} \langle u(0) u(x) | \xi \rangle \Big|_{x=0} \\ &\rightarrow \xi^2 \frac{\langle u \frac{\partial u}{\partial x} \rangle \langle \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \rangle}{\left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle^2} = 0, \end{aligned}$$

where the zero comes from the assumed statistical homogeneity. Using now expression (27) with $m = 1$ we have, using also the condition of homogeneity,

$$\begin{aligned} \left\langle \frac{\partial^3 u}{\partial x^3} \middle| \xi \right\rangle &= \frac{\partial^3}{\partial x^3} \langle u(x) | \xi \rangle \Big|_{x=0} \\ &\rightarrow -\xi \frac{\left\langle \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right\rangle}{\left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle}. \end{aligned}$$

so that, calling $\kappa \equiv \nu \left\langle (\partial^2 u / \partial x^2)^2 \right\rangle \left\langle (\partial u / \partial x)^2 \right\rangle^{-1} = \varepsilon^{-1} \left\langle (\nu \partial^2 u / \partial x^2)^2 \right\rangle$, we have for the statistically stationary case, using also the expression (45) for the forcing,

$$\frac{\partial}{\partial \xi} \left[(\xi^2 + \kappa \xi) P - \varepsilon \frac{\chi''(0)}{L_c^2} \frac{\partial P}{\partial \xi} \right] = 0,$$

and analogously to the velocity difference case we obtain for $\xi > 0$

$$\ln P = \frac{L_c^2}{\chi''(0)} \left(\frac{\xi^3}{3\varepsilon} + \frac{\kappa \xi^2}{2\varepsilon} \right),$$

and, for $\xi < 0$,

$$P \sim \xi^{-2}.$$

All these results for the Burgers equation reproduce well the behavior obtained theoretically and numerically in the references cited above.

IV. Conclusions

We have presented a simple closure for the asymptotic pdf evolution equation of rather generic turbulence problems. We have presented results for the simplest possible problems in turbulence, for which a rather detailed knowledge exists. For the paradigmatic problem of Navier-Stokes turbulence the application of the formalism is more involved, and has been applied in a particular version in,²² where very reasonable results were obtained, although comparison with detailed pdf 's is more difficult.

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